Five Year Integrated M. Sc. Examination, 2023
Semester-V
Course: MT-3-5-2
(Analysis I)

Time: Four Hours **Full Marks: 80**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Answer ***any eight*** questions.

1. (a) When does a sequence of functions on a set converge pointwise? Explain with examples. [1+3]
   (b) Let $f_n : [0, \infty) \longrightarrow \mathbb{R}$ be given by $f_n(x) = \frac{nx}{1+nx}$. Show that the sequence $\{f_n\}$ is pointwise convergent on $[0, \infty)$. Find the limit function of $\{f_n\}$. [4+2]
2. (a) Define the functions $f_n : [0, 1] \longrightarrow \mathbb{R}$ by $f_n(x) = 1 - \frac{x^n}{n}$. Show that $\{f_n\}$ is uniformly convergent on $[0, 1]$. [5]
   (b) Let a sequence $\{f_n\}$ of continuous functions on a set $A \subset \mathbb{R}$ converge uniformly to a function f. Prove that f is continuous on A. [5]
3. (a) What is Weierstrass' M-test for uniform convergence of a sequence of functions? Use it to examine the uniform convergence of the series $\displaystyle\sum_{1}^{\infty} \frac{1}{n^5 + n^4 x^2}$ on $\mathbb{R}$. [1+4]
   (b) Describe Dirichlet's test and hence show that the series $\displaystyle\sum_{1}^{\infty} (-1)^n x^n (1 - x)$ converges uniformly on $[0, 1]$. [1+4]
4. (a) Prove or disprove: The uniform convergence of a series of functions is sufficient to ensure the validity of term-by-term differentiation of the series on a closed and bounded interval. [7]
   (b) Determine if the series $\displaystyle\sum_{1}^{\infty} \frac{\sin(2^n x)}{n^2}$ is uniformly convergent on $\mathbb{R}$. [3]
5. (a) If a power series $\displaystyle\sum_{0}^{\infty} a_n x^n$ converges for $x = x_1$, prove that it converges absolutely for all $x \in \mathbb{R}$ satisfying $|x| < |x_1|$. [5]
   (b) Find the radius and interval of convergence of the power series $\displaystyle\sum_{0}^{\infty} \frac{n^n x^n}{n! 2^n}$. [5]
6. (a) Prove that a power series $\displaystyle\sum_{0}^{\infty} a_n x^n$ with radius of convergence $R(> 0)$ is uniformly convergent on $[-b, b]$, where $0 < b < R$. [6]
   (b) Let $\displaystyle\sum_{0}^{\infty} a_n x^n$ be a power series with radius of convergence $R(> 0)$ and let $f(x)$ be its sum function on $(-R, R)$. Show that $f^{(n)}(0) = n! a_n$, for $n = 0, 1, 2, \cdots$. [4]
7. (a) What do you mean by an improper integral and its convergence in different cases? Explain in detail. [3+3]
   (b) Show that the improper integral $\displaystyle\int_0^2 \frac{dx}{\sqrt{x(2 - x)}}$ is convergent. [4]

8. (a) Prove that the improper integral $\int_a^\infty \frac{dx}{x^n}$ (where $a > 0$) is convergent if and only if $n > 1$. [6]

(b) When is an improper integral $\int_{-\infty}^{\infty} f dx$ called absolutely convergent and conditionally convergent? Give examples. [4]

9. (a) Describe when a bounded function $f : [a, b] \longrightarrow \mathbb{R}$ becomes Riemann integrable. [4]

(b) For any partition P of a closed and bounded interval $[a, b]$ and a bounded function $f : [a, b] \longrightarrow \mathbb{R}$, show that $L(P, f) \leq U(P, f)$. [6]

10. (a) Let the function $f : [a, b] \longrightarrow \mathbb{R}$ be bounded on $[a, b]$. Prove that f is Riemann integrable on $[a, b]$ if and only if for each $\epsilon > 0$, there exists a parttition P of $[a, b]$ such that $U(P, f) - L(P, f) < \epsilon$. [8]

(b) Define a refinement and the norm of a partition of $[a, b]$. [2]

11. (a) Prove or disprove: A monotone function $f : [a, b] \longrightarrow \mathbb{R}$ is Riemann integrable on $[a, b]$. [5]

(b) A function $f : [0, 1] \longrightarrow \mathbb{R}$ is given by $f(x) = \begin{cases} x & \text{if} \quad x \in [0,1] \cap \mathbb{Q} \\ 0 & \text{if} \quad x \in [0,1] \cap \mathbb{Q}^c. \end{cases}$ Find the lower and upper Riemann integrals of f on $[0, 1]$ and determine if f is Riemann integrable on $[0, 1]$. [5]

12. (a) If a bounded function $f : [a, b] \longrightarrow \mathbb{R}$ is continuous on $[a, b]$ except for a finite number of points in $[a, b]$, show that f is Riemann integrable on $[a, b]$. [7]

(b) Let $f(x) = [x]$, $x \in [-2, 2]$. Determine if f is Riemann integrable on $[-2, 2]$. [3]